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RESEARCH NOTE

The similitude equation in magnetotelluric inversion

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SUMMARY

The similitude equation for electromagnetic induction represents an exact integral relationship between the conductivity model and field measurements, and has been suggested as a basis for the inversion of magnetotelluric data. In this note, inversion of the similitude equation is compared to linearized inversion and found to be inadequate in that it implicitly neglects first-order terms.

Key words: inversion, linearization, magnetotellurics similitude equation.

INTRODUCTION

The magnetotelluric (MT) method uses surface measurements of natural electromagnetic fields to investigate the conductivity distribution of the Earth. Determining the conductivity model from a set of MT responses is a non-linear inverse problem. Many successful approaches to inverting MT responses are based on linearization of the data equations. Linearized methods are particularly useful when they are used to generate solutions which minimize a given functional of the model. Models of different character can be constructed by minimizing different functionals. Gómez-Treviño (1987) presented an alternative formulation for the MT inverse problem. Using scaling properties of Maxwell's equations, he derived an exact, non-linear integral equation, known as the similitude equation, relating the conductivity model to field measurements. Gómez-Treviño suggested that the similitude equation should be investigated as the basis for an inversion procedure which parallels linearization in generality. This note compares the similitude equation to linearization for MT inversion.

LINEARIZED INVERSION

Linearization has proved to be a practical and useful method for solving the non-linear MT inverse problem. In the linearized approach, the MT response R for the true model $\sigma(z)$ at period T is expanded about an arbitrary starting model $\sigma_0(z)$ according to

$$R(\sigma, T) = R(\sigma_0 + \delta\sigma, T) = R(\sigma_0, T) + \int_0^\infty G(\sigma_0, T, z)\,\delta\sigma(z)\,dz + R_\sigma, \quad (1)$$

where R_σ represents the remainder term of the first-order expansion in conductivity. Assuming R is Fréchet differentiable, G is known as the Fréchet kernel, and the remainder term is second order in $\delta\sigma$. Neglecting the remainder term yields an approximate linear equation

$$\delta R(\sigma, \sigma_0, T) = \int_0^\infty G(\sigma_0, T, z)\,\delta\sigma(z)\,dz, \quad (2)$$

where $\delta R(\sigma, \sigma_0) = R(\sigma) - R(\sigma_0)$. Substituting $\delta\sigma(z) = \sigma(z) - \sigma_0(z)$ in (2) leads to a linear expression relating known quantities ('modified' responses) to a linear functional of the model:

$$\delta R(\sigma, \sigma_0, T) + \int_0^\infty G(\sigma_0, T, z)\sigma_0(z)\,dz = \int_0^\infty G(\sigma_0, T, z)\sigma(z)\,dz. \quad (3)$$

This expression is in the form of a Fredholm integral equation of the first kind, and standard methods of linear inverse theory may be used to solve for a model that minimizes some functional of $\sigma(z)$. Since higher order terms have been neglected, this procedure must be repeated iteratively until an acceptable model is achieved.

This formulation provides considerable flexibility: solutions of different character may be constructed by minimizing different functionals of the model. Oldenburg (1983) and Dosso & Oldenburg (1989) have used this method to construct extremal models which maximize or minimize a box-car average of the conductivity over a specified region. These extremal models provide upper and lower bounds for the conductivity averages and may be used to appraise model features of interest. Constable, Parker & Constable (1987), Smith & Booker (1988) and Dosso & Oldenburg (1989) have also used the linearized equation (3) to construct minimum-structure conductivity models from MT measurements. The iterative linearized inversion procedure is essentially Newton's method for operators (e.g. Milne 1980, chapter 4), so convergence is generally guaranteed provided the starting model is sufficiently close to an acceptable solution, and quadratic convergence can be expected when the method converges.

INVERSION USING THE SIMILITUDE EQUATION

Gómez-Treviño (1987) derived the similitude equation for MT in terms of the apparent conductivity σ_a as response. In order to compare the similitude approach to linearization, the similitude equation is derived here for the MT response $R = B/E$, where B and E represent orthogonal components of the magnetic and electric fields measured at the surface of the Earth. This derivation also serves to illustrate Gómez-Treviño's approach. The scaling properties of the electric and magnetic fields are well known and follow directly from Maxwell's equations: for a scalar k

$$E(k\sigma, kT, z) = \frac{1}{k} E(\sigma, T, z), \tag{4}$$

$$B(k\sigma, kT, z) = B(\sigma, T, z). \tag{5}$$

Thus, the R response scales according to

$$R(k\sigma, kT) = \frac{B(k\sigma, kT, 0)}{E(k\sigma, kT, 0)} = kR(\sigma, T). \tag{6}$$

If $k = 1 + h$, (6) becomes

$$R(\sigma + h\sigma, T + hT) = (1 + h)R(\sigma, T). \tag{7}$$

The quantities $h\sigma$ and hT may be thought of as perturbations in conductivity and period which are simply a scaling of the original values. The perturbation in the response δR is given by

$$\delta R(h\sigma, hT) = R(\sigma + h\sigma, T + hT) - R(\sigma, T). \tag{8}$$

Combining (7) and (8) leads to

$$\delta R(h\sigma, hT) = hR(\sigma, T). \tag{9}$$

However, δR may also be expressed in terms of an expansion about T and σ:

$$\delta R(h\sigma, hT) = hT \frac{\partial R(\sigma, T)}{\partial T} + \int_0^\infty G(\sigma, T, z) h\sigma(z)\, dz + R_T + R_\sigma, \tag{10}$$

where R_T and R_σ are (second-order) remainder terms for the first-order expansion of R about σ and T. For the response $R = B/E$, the Fréchet kernel is given by $G(\sigma_0, T, z) = -\mu_0[E(\sigma_0, T, z)/E(\sigma_0, T, 0)]^2$ (Oldenburg 1979). Substituting for δR from (9) and dividing by h, (10) may be written as

$$R(\sigma, T) = T \frac{\partial R(\sigma, T)}{\partial T} + \int_0^\infty G(\sigma, T, z)\sigma(z)\, dz + \frac{1}{h} R_T + \frac{1}{h} R_\sigma. \tag{11}$$

In the limit as $h \to 0$, $R_T/h \to 0$ and $R_\sigma/h \to 0$ and (11) becomes

$$R(\sigma, T) - T \frac{\partial R(\sigma, T)}{\partial T} = \int_0^\infty G(\sigma, T, z)\sigma(z)\, dz. \tag{12}$$

(12) is the similitude equation for R. It is an exact expression relating field measurements and their derivatives to the conductivity model and makes no allusion to any perturbation or starting model; however, the relationship is non-linear in σ. By comparison, the linearized equation (3) is an approximate expression relating responses to a linear functional of σ which is accurate to second order.

Gómez-Treviño (1987) suggests that since the similitude equation relates the response to the model exactly, inversion algorithms could be based on this formulation rather than on a linearized approximation. The difficulty is that the kernel function in (12) is evaluated at the (unknown) model $\sigma(z)$. Gómez-Treviño suggested approximating $G(\sigma, T, z)$ with $G(\sigma_0, T, z)$ in (12), and solving the resulting linear equation for σ in an iterative manner. Although he does not consider the problem in detail, Gómez-Treviño suggested this approach could form a general inversion procedure for MT.

There are several practical difficulties with an inversion algorithm based on the similitude equation. The modified data involve both the measured response $R(\sigma, T)$ and its derivative with respect to period $\partial_T R(\sigma, T)$. In order to evaluate $\partial_T R(\sigma, T)$, the response $R(\sigma, T)$ must be measured at closely spaced periods. Even if this is done, estimating numerical derivatives is prone to error and therefore the final data are subject to errors that are potentially much larger than the measurement errors in $R(\sigma, T)$ alone. In addition, if the inversion algorithm converges, all that is required is that the similitude data $R(\sigma, T) - T\, \partial_T R(\sigma, T)$ are reproduced. This does not ensure that either $R(\sigma, T)$ or $\partial_T R(\sigma, T)$ are reproduced individually.

A more fundamental difficulty with inversion via the similitude equation involves the approximation of $G(\sigma, T, z)$ with $G(\sigma_0, T, z)$ in (12). This approximation is equivalent to neglecting an error term R_s in much the same way that expansion (1) is linearized by neglecting the remainder term R_σ. To illustrate this, the similitude equation (12) can be written

$$R(\sigma, T) - T \frac{\partial R(\sigma, T)}{\partial T} = \int_0^\infty G(\sigma_0, T, z)\sigma(z)\, dz + R_s. \tag{13}$$

The error term R_s, which is neglected to invert (13), is given by

$$R_s = \int_0^\infty [G(\sigma, T, z) - G(\sigma_0, T, z)]\sigma(z)\, dz. \tag{14}$$

To determine the magnitude of the similitude error term R_s, consider the following analysis. Subtract from the similitude equation (12) an identical expression evaluated at σ_0 rather than σ. This leads to

$$\delta R - T \frac{\partial(\delta R)}{\partial T} = \int_0^\infty G(\sigma, T, z)\sigma(z)\, dz - \int_0^\infty G(\sigma_0, T, z)\sigma_0(z)\, dz. \tag{15}$$

After some algebra, (15) may be rearranged to give

$$R_s = \delta R - \int_0^\infty G(\sigma_0, T, z)\, \delta\sigma(z)\, dz - T \frac{\partial(\delta R)}{\partial T}. \tag{16}$$

But $\delta R - \int_0^\infty G(\sigma_0, z)\, \delta\sigma(z)\, dz = R_\sigma$ according to (1), so

(16) becomes

$$R_s = R_\sigma - T\frac{\partial(\delta R)}{\partial T}. \tag{17}$$

From (1) it follows that

$$T\frac{\partial(\delta R)}{\partial T} = T\int_0^\infty \frac{\partial G(\sigma_0, T, z)}{\partial T}\,\delta\sigma(z)\,dz + T\frac{\partial R_\sigma}{\partial T}, \tag{18}$$

and so

$$R_s = -T\int_0^\infty \frac{\partial G(\sigma_0, T, z)}{\partial T}\,\delta\sigma(z)\,dz + R_\sigma - T\frac{\partial R_\sigma}{\partial T}. \tag{19}$$

Using (19), the similitude equation (13) can be written as

$$\begin{aligned} R(\sigma, T) - T\frac{\partial R(\sigma, T)}{\partial T} &= \int_0^\infty G(\sigma_0, T, z)\,dz \\ &\quad - \int_0^\infty T\frac{\partial G(\sigma_0, T, z)}{\partial T}\,\delta\sigma(z)\,dz \\ &\quad + R_\sigma - T\frac{\partial R_\sigma}{\partial T}. \end{aligned} \tag{20}$$

In order to illustrate the order in $\delta\sigma$ of the terms, (20) can be written as

$$\begin{aligned} &R(\sigma, T) - T\frac{\partial R(\sigma, T)}{\partial T} - \int_0^\infty G(\sigma_0, T, z)\sigma_0(z)\,dz \\ &= \int_0^\infty G(\sigma_0, T, z)\,\delta\sigma(z)\,dz - \int_0^\infty T\frac{\partial G(\sigma_0, T, z)}{\partial T}\,\delta\sigma(z)\,dz \\ &\quad + R_\sigma - T\frac{\partial R_\sigma}{\partial T}. \end{aligned} \tag{21}$$

The quantities on the left side of (21) may be considered modified data. The first term on the right side is the linear functional to be inverted, expressed here in terms of $\delta\sigma$; the remaining three terms comprise the error term R_s. The third and fourth terms on the right are second order in $\delta\sigma$, but the second term is first order in $\delta\sigma$. If R_s is neglected in (13) then this first-order term is not included in the inversion. Writing the similitude equation in this form emphasizes that the linear functional to be inverted and the error term are of the same order in $\delta\sigma$, and any inversion scheme based on neglecting this error term is ill-founded. Whether the model produced by such an inversion is an improvement on σ_0 depends on the relative size of the linear functionals of $\delta\sigma$ in (21). At best, an iterative inversion algorithm based on the similitude equation might exhibit linear convergence; however, convergence is not guaranteed, even as $\sigma_0 \rightarrow \sigma$.

A natural question to investigate is whether the first-order error term can be included in the inversion rather than neglected. In order to include this term, the similitude equation (20) may be written to first order as

$$\begin{aligned} &R(\sigma, T) - T\frac{\partial}{\partial T}\left[R(\sigma, T) - \int_0^\infty G(\sigma_0, T, z)\,\delta\sigma(z)\,dz\right] \\ &= \int_0^\infty G(\sigma_0, T, z)\sigma(z)\,dz. \end{aligned} \tag{22}$$

Since $R(\sigma, T) - \int_0^\infty G(\sigma_0, T, z)\,\delta\sigma(z)\,dz = R(\sigma_0, T)$ (to first order) according to (2), and $T\,\partial_T R(\sigma_0, T) = R(\sigma_0, T) - \int_0^\infty G(\sigma_0, T, z)\sigma_0(z)\,dz$ according to (12), (22) reduces immediately to the linearized equation

$$\begin{aligned} &\delta R(\sigma, \sigma_0, T) + \int_0^\infty G(\sigma_0, T, z)\sigma_0(z)\,dz \\ &= \int_0^\infty G(\sigma_0, T, z)\sigma(z)\,dz. \end{aligned} \tag{3}$$

This is not a surprising result. The Fréchet derivative of the response R with respect to the conductivity is defined to be a linear functional which, when applied to the conductivity perturbation, produces (to first order) the response perturbation (e.g. Milne 1980, chapter 4). The Fréchet derivative is unique, therefore the linearized equation represents the only linear expression which relates δR to $\delta\sigma$ [or, using (3), to σ itself] which is accurate to first order in $\delta\sigma$. Any attempt to devise a linear relationship between R and σ which is accurate to first order must reduce to the linearized equation for R.

DISCUSSION

In this note the similitude equation is compared to linearization as a basis for the inversion of MT responses. The similitude equation is shown to be inappropriate for general inversion in that its application implicitly neglects a first-order error term. Before the analysis presented in this note was carried out, an inversion algorithm based on the similitude equation was implemented and tested for synthetic cases where $R(\sigma, T)$ and $\partial_T R(\sigma, T)$ could be computed accurately. For the special case when the true and starting models consisted of half-spaces of constant conductivity, the algorithm generally converged to the true model. However, for models with any appreciable structure (even simple two-layer models), the algorithm did not converge. Thus it would seem both in theory and in practice that the similitude equation is not an appropriate basis for a general inversion algorithm.

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