An adaptive mesh method for electromagnetic inverse problems

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SUMMARY

Electromagnetic data can be used to recover the electrical conductivity of the subsurface of the earth. In order to extract quantitative value from an exploration survey, the data must be inverted. When working with traditional 3D regular rectangular meshes, the number of cells required to accurately discretize the earth can be prohibitively large. This is especially true for large irregularly shaped surveys or regions of severe topography. Using an adaptive mesh structure, such as an OcTree, high model resolution can be given where it is needed, while cells can be made coarse where possible. We develop an electromagnetic inverse algorithm on a semi-structured OcTree mesh. The algorithm is applied to a synthetic example of controlled source frequency domain and to a field ZTEM data example.

INTRODUCTION

Electromagnetic (EM) techniques can be used to explore for 3D targets inside the earth. They are commonly applied to mineral and hydrocarbon exploration, as well as to environmental studies. Depending on the target size, geometry, and conductivity, appropriate surveys are designed with sources used to excite the earth, and receivers to record the resulting fields. The goal of the inverse problem is to recover the earth's conductivity given the measurements of these fields. The fields can be recorded either in the frequency or the time domain. In this paper we concentrate on frequency domain problems, but our approach can easily be extended into the time domain.

Given the wide applicability of EM techniques, solving EM inverse problems is now common in many geophysical applications. Many algorithms and codes exist; most based on the numerical solution of partial differential equations (PDE) and use finite difference, volume, or elements (Newman and Alumbaugh, 1997a,b; Haber et al., 2000a). The disadvantage of most current PDE based methods is that they use stretched orthogonal meshes (Newman and Alumbaugh, 1997b; Haber et al., 2000a). Using orthogonal meshes makes the codes easy to program and work with; however, these meshes can lead to over-discretization in areas that the fields change slowly. In order to have accurate modeling, it is possible to use mesh refinement. The idea is not new and is a central theme in numerical PDEs (Edwards, 1996; Trottenberg et al., 2001; Wang et al., 2002; Eriksson et al., 1995). The main idea is to have large cells where the fields have no significant changes, and to have much smaller cells in areas that the fields change rapidly. The concept is commonly used in the context of the forward problem however, for inverse problems the methodology is relatively new (Haber et al., 2007; Bangerth, 2002; Becker et al., 2000).

When using adaptive meshes, two possible approaches have been suggested. One popular approach is to discretize the domain using a mesh made of tetrahedra and use an unstructured mesh for the simulation and inversion. Our recent experience is that although the approach is popular in many science and engineering applications, it requires some nontrivial tools and is difficult to work with in practice. First, a reliable mesh generator is needed. We have found that some popular mesh generators used to describe geologic structures (Mallet et al., 1989) lead to meshes with large aspect ratios between elements and hence ill-conditioned matrices. Second, unstructured meshes lead to unstructured sparse matrices which are difficult to deal with. Finally, manipulating and viewing conductivity structures obtained on unstructured meshes is a nontrivial matter.

A second approach is to use structured orthogonal rectangular meshes and locally refine cells when needed. This strategy leads to regular meshes that are locally refined. If the local refinement is regular then this naturally leads to an OcTree structure (Hjaltason and Samet, 2002). While the approach is less economic compared to an unstructured mesh, we have found that the extra over-head is small. This additional overhead is not an impediment because the approach leads to structured matrices with predictable patterns and no ill-conditioning. An additional benefit of these mesh structures is that they are easy to manipulate and view (Haber et al., 2006).

In this work we combine OcTree meshes, effective linear algebra and optimization algorithms (Haber et al., 2007; Horesh and Haber, 2011; Haber and Heldmann, 2007) in order to solve large scale EM inverse problems. We demonstrate the advantages of our techniques on both a synthetic ground loop frequency domain survey as well as an airborne ZTEM survey.

THE FORWARD PROBLEM

In this section we briefly review the work of Haber and Heldmann (2007); Horesh and Haber (2011); Chevalier et al. (1997) and discuss the discretization of Maxwell's equations on an OcTree mesh. Maxwell's equations in the frequency domain can be written as

$$\nabla \times \vec{E} = -i\omega\mu\vec{H} \tag{1a}$$

$$\nabla \times \vec{H} = \sigma\vec{E} + \vec{q} \tag{1b}$$

$$\widehat{n} \times \vec{H} = 0 \tag{1c}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields, μ is the magnetic permittivity, σ is the (possibly complex) conductivity, ω is the angular frequency and $\vec{q}$ stands for an external current density source. Here we chose boundary conditions on the tangential components of $\vec{H}$ but other appropriate boundary conditions can equally be prescribed. For our applications we consider the quasi-static case and thus the system can

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be viewed as a diffusion equation (Weaver, 1994; Ward and Hohmann, 1988). There are three main difficulties when solving the system. First, the conductivity σ varies over several orders of magnitude. Second, due to the nontrivial null-space of the **curl** operator the resulting linear systems are highly ill-conditioned. Finally, the fields may vary rapidly in the vicinity of sources and become rather smooth afar, requiring very high resolution at some regions of the domain.

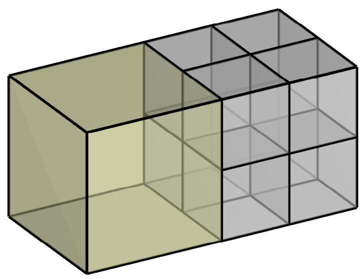

Figure 1: OcTree mesh

To discretize Maxwell's equations we consider a fine underlying orthogonal mesh of size $2^{m_1} \times 2^{m_2} \times 2^{m_3}$ with grid size h. Our 3D grid is composed of m square cells of different sizes. The length size of each cell may carry powers of 2 multiplied by h. To make the data structure easier and the discretization more accurate only a factor of 2 in length size between adjacent cells is permitted. An example of a small 3D OcTree grid is displayed in Figure 1.

Discretization of the equations

To discretize Maxwell's equations we extend Yee's method (Yee, 1966) and use a staggered mesh on OcTrees. Here we briefly present the extension. For a detailed discussion see Haber and Heldmann (2007); Horesh and Haber (2011); Lipnikov et al. (2004). Let $\mathbf{e}$ be a discrete grid function of the electric field and $\mathbf{h}$ be a discrete grid function of the magnetic field. Without loss of generality we choose to discretize $\mathbf{e}$ on cell edges and $\mathbf{h}$ on cell faces. The discretizations of the face-divergence, the nodal-gradient and the edge-curl are natural, that is, they are obtained by considering the primal definition of the operators. The discretization of their adjoint, that is, the curl of a face valued function or the divergence of an edge vector are not. However, in order to discretize Maxwell's equations the adjoint operators are needed. There are two possible approaches to deal with this difficulty. First, it is possible to use finite difference approximations to the adjoint operators that are of any desired accuracy (Horesh and Haber, 2011). This approach leads to second order discretization with a loss of symmetry. A different approach that we take here following Haber and Heldmann (2007); Lipnikov et al. (2004), is to use a weak formulation of the equations obtained by integration by parts (for details see Lipnikov et al. (2004)). The idea is to use appropriate vector functions $\vec{W}$ and $\vec{F}$ and write Maxwell's equations in weak form

$$(\nabla \times \vec{E}, \vec{F}) = -i\omega(\mu\vec{H}, \vec{F}) \tag{2a}$$
$$(\vec{H}, \nabla \times \vec{W}) = (\sigma\vec{E}, \vec{W}) + (\vec{q}, \vec{W}) \tag{2b}$$

where $(\cdot,\cdot)$ is the usual inner product in L_2 and Equation (2b) is obtained by integration by parts assuming $\vec{n} \times \vec{H} = 0$. Let $\mathbf{f}$ and $\mathbf{w}$ be discrete analogs of the continuous variables $\vec{F}$ and $\vec{W}$. If we choose to discretize $\mathbf{e}$ on the edges and $\mathbf{h}$ on the faces then $\mathbf{f}$ is a face grid function and $\mathbf{w}$ is an edge grid function. The discretization is reduced to using an appropriate quadrature formula to represent the inner product.

To that end we define the averaging matrices A_f and A_e as the matrices that average from faces and edges into cell centers and let $\mathbf{v}$ be a vector such that $\mathbf{v}_i$ is the volume of the i^{th} cell. Furthermore, with some abuse of notations, let μ and σ be the magnetic susceptibility and the electric conductivity of each OcTree cell. Using the **curl** matrix and the averaging operators the weak form system is written as

$$C\mathbf{e} = -i\omega \mathsf{diag}(A_f^\top(\mu \odot \mathbf{v}))\,\mathbf{h} \tag{3a}$$
$$C^\top\mathbf{h} = \mathsf{diag}(A_e^\top(\sigma \odot \mathbf{v}))\,\mathbf{e} + \mathbf{q} \tag{3b}$$

where $\odot$ is the Hadamard product. Eliminating $\mathbf{h}$ from the system we obtain an equivalent second order system

$$C^\top \left(\mathsf{diag}(A_f^\top(\mu \odot v))\right)^{-1} C\mathbf{e} + i\omega \mathsf{diag}(A_e^\top(\sigma \odot v))\,\mathbf{e} = i\omega\mathbf{q} \tag{4}$$

System (4) is the system solved for the electric field given sources and conductivity.

Discretization of sources

Different approaches are taken for controlled and natural sources. Controlled sources are discretized directly on the OcTree mesh while natural sources are discretized using a secondary field approach. To discretize controlled sources we use a direct approach and assume that the sources align with the OcTree mesh. The source is assumed to be a delta function on the edge and the vector $\mathbf{q}$ in (4) is computed by assuming the integration of a delta function over the appropriate edge. Natural sources are treated by secondary field approach. We have used a 1D code that computes the response of the earth due to a plain wave source and then solve (4) with a new right hand side. It is easy to verify that this approach is equivalent to a primary/secondary approach.

Solution of the linear system

The discretization of Maxwell's equations leads to large linear systems to be solve. There are two main cases to consider. First, when the number of sources is small (for example in the case of natural sources) then iterative methods are used. Second, when the number of sources is large (for example in controlled source airborne) then a direct method is used.

INVERSE PROBLEM

Inverse problem formulation

We assume to have n_s sources $Q = i\omega[\mathbf{q}_1, \ldots, \mathbf{q}_{ns}]$. Let $A(\mathbf{m})$ be the discrete linear system obtained by the discretization of Maxwell's equations with $\mathbf{m} = \log(\sigma)$. Finally, let $P^\top : n_e \to n_r$ be a matrix that projects the edge base solution to n_r receivers (for details see Haber et al. (2000b)). Then the simulated data is an $n_r \times n_s$ matrix can be written as

$$D(\sigma) = P^\top A(\mathbf{m})^{-1} Q$$

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The datum D_{ij} represents the i^{th} data due to the j^{th} source. Let D^{obs} be the observed data, that is the data that is collected in the field. Let Σ be a matrix that holds the standard deviation of each datum, that is Σ_{ij} is the inverse standard deviation of D^{obs}_{ij}. The misfit between the observed and the predicted data can be written as

$$\mathscr{J}_{\text{mis}}(\mathbf{m}) = \frac{1}{2}\|\Sigma \odot (P^\top A(\mathbf{m})^{-1}Q - D^{\text{obs}}\|_F^2$$

where $\| \, . \, \|_F$ is the matrix Forbinious norm. Since the problem does not admit stable solutions, regularization is needed. Here we use the so called "smoothness" regularization. To this end we define the discrete cell-center gradient matrix G_c and use the regularization

$$\mathscr{J}_{\text{reg}}(\mathbf{m}) = \frac{1}{2}\|VG_c(\mathbf{m} - \mathbf{m}_{\text{ref}}\|^2$$

where the matrix V is a diagonal matrix with the cell volumes, and $\mathbf{m}_{\text{ref}}$ is a reference model chosen to represent a priori information about the model. The model is obtained by minimizing a linear combination of the misfit and the regularization

$$\begin{aligned} \min_{\mathbf{m}} \quad & \mathscr{J}_{\text{mis}}(\mathbf{m}) + \alpha \mathscr{J}_{\text{reg}}(\mathbf{m}) \\ \text{s.t} \quad & \mathbf{m}_L \le \mathbf{m} \le \mathbf{m}_H \end{aligned} \tag{5}$$

where α is a regularization parameter and $\mathbf{m}_L$ and $\mathbf{m}_H$ are vectors of upper and lower bounds that are set based on geological a priori information.

Optimization algorithms

To solve the inverse problem we employ two main algorithms. The first is a simple Gauss-Newton method coupled with a "cooling" strategy for the regularization parameter and the second is a Gauss-Newton method coupled with an iterated Tikhonov regularization method. To this end, the gradient of the misfit can be written as

$$\vec{\nabla}\mathscr{J}_{\text{mis}} = \sum_j G_j(\mathbf{m})^\top A(\mathbf{m})^{-\top} P \left(\Sigma_j^2 \odot (P^\top A(\mathbf{m})^{-1}Q_j - D_j^{\text{obs}})\right)$$

where $(.)_j$ is the j^{th} column of the matrix and

$$G_j = \frac{\partial (A(\mathbf{m})\mathbf{u}_j)}{\partial \mathbf{m}}.$$

The computation of the gradient requires the solution of the forward and adjoint problems for all sources and data residuals. Defining the sensitivity matrix

$$J = \sum_j P^\top A^{-1} G_j$$

the Gauss Newton step can be written as

$$(J^\top J + \alpha\delta^2 R)s = -\vec{\nabla}\mathscr{J}_{\text{mis}}(\mathbf{m}) - \vec{\nabla}\mathscr{J}_{\text{reg}}(\mathbf{m} - \mathbf{m}_{\text{ref}}).$$

The Gauss Newton system is then solved using conjugate gradient where only matrix-vector product are computed (see Newman and Alumbaugh (1997a) for further discussion).

Two classes of algorithms can be used to obtain a suitable data misfit. First, it is possible to use Tikhonov regularization where we fix $\mathbf{m}_{\text{ref}}$ and gradually decrease α until a desirable misfit is obtained. The second approach is to fix α and iteratively update $\mathbf{m}_{\text{ref}}$, which leads to the so-called iterated Tikhonov regularization (Hanke and Groetsch, 1998). In our numerical experiments we found that the iterated Tikhonov regularization performs slightly better due to better conditioning of the Gauss-Newton iteration while leading to approximately similar results.

EXAMPLES

Frequency Domain Loop Example

In this example a synthetic model was created based on drill hole information. The synthetic model has discrete conductors contained within a dipping zone of highly variable conductivity. Locating these conductive targets is made more difficult by overlaying folded units, one of which is conductive, the other resistive. Given this information, a 3D frequency domain ground loop EM survey with 25 transmitter loops and an array of 64 receivers was designed. The configuration of the survey was such that each transmitter (12 frequencies logarithmically spaced between 0.1 Hz and 1000 Hz) was used one at a time, with the vertical component of the magnetic fields recorded at each of the 64 receivers. In total this survey geometry generated 19,200 data which after synthetic noise was added, were inverted. Despite the difficulties of recovering the targets beneath the overlying conductive units, the results show that the survey is able to detect these structures within the dipping unit down to a depth of approximately 500m. In addition to this, the inversion result (Figure 2) indicates that both the dipping unit and the folded unit can be accurately recovered.

ZTEM Field Dataset

Finally we consider a ZTEM field dataset. The Z-Axis Tipper Electromagnetic (ZTEM) method (Holtham and Oldenburg, 2010; Lo and Zang, 2008) is an airborne natural source EM technique developed by Geotech Ltd. that is effective at mapping large-scale geologic structures. During a ZTEM survey, a helicopter is used to measure the vertical magnetic field over the surface of the earth. The measured data (typically between 30-720 Hz depending on signal strength) relate the measured vertical magnetic field recorded at the helicopter, to the horizontal magnetic field measured at a ground based reference station. For this 2000 line-km ZTEM survey, 6 frequencies of data were collected over two separate flights blocks. Both blocks contained extreme topographic relief and the survey geometry (Figure 3) was non-rectangular for the block we consider here. This topography and survey flight line geometry would make discritizing the earth very inefficient using a regular rectangular mesh.Using an OcTree mesh that adapts the cell size when necessary can greatly reduce the number of cells in the model (for this example the underlying rectangular mesh had 16.7 million cells while the refined OcTree mesh only had 1.8 million cells). Figure 4 shows the OcTree mesh used to invert the data. The blue cells are fine cells near the data and topographic surface. Away from the area of interest, the cell sizes adapts and quickly becomes large. Figure 5 shows the resistive and conductive structures from the inverted model.

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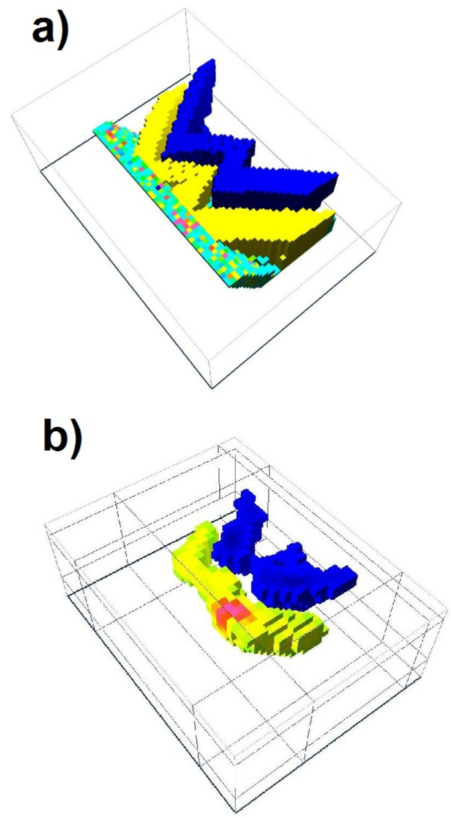

Figure 2: a) True conductivity model. b) Recovered conductivity model. The W shape of the folded units is clearly discernible in the recovered model. The conductive targets contained within the dipping structure are visible as bright red in the recovered model.

CONCLUSIONS

Traditionally large field EM inversions have been performed in 1D or 2D because of the difficultly of inverting large datasets in 3D. Although computationally simple, working in 1D or 2D can lead to incorrect interpretations and inefficient use of resources since real geologic targets are complex and 3D in nature. In this paper we show how electromagnetic inverse problems can be solved on semi-structured OcTree meshes. Going away from regular rectangular meshes and instead working with semi-structured meshes allows us to more accurately model geologic structures using fewer cells which ultimately results in faster codes and makes inverse problems that were previously too large to be solved in 3D possible. We have tested the algorithm on both a synthetic frequency domain controlled source survey as well as a ZTEM field dataset.

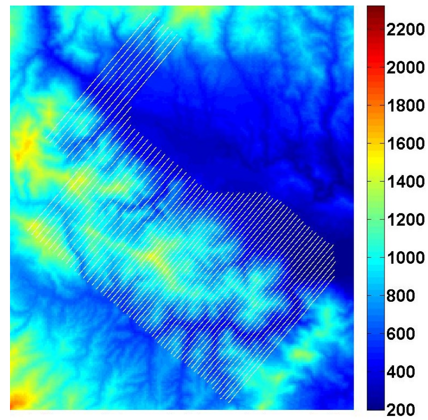

Figure 3: ZTEM flight lines and topography in meters. Because of the irregular flight block geometry and extreme topography, using semi-structured meshes are vital to limit the number of wasted cells.

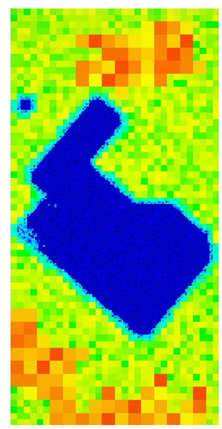

Figure 4: OcTree mesh highlighting the adaptive nature of the mesh. The blue cells are the finer cells which are required near the data locations and where the topography is rapidly changing. In other regions of the mesh the cells can expand to greatly reduce the number of cells. This OcTree mesh contained 1.8 million cells, while the underlying regular mesh contained 16.7 million cells.

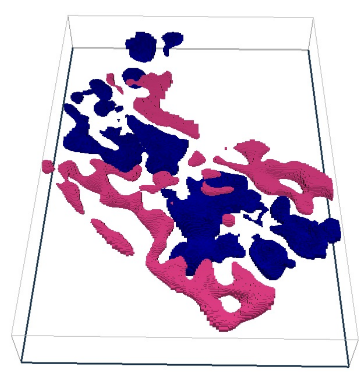

Figure 5: ZTEM inversion showing the more conductive units (pink) and the more resistive units (blue).

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EDITED REFERENCES

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