A Multiscale Method for Fast Electromagnetic Simulations

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SUMMARY

Simulating low-frequency electromagnetic fields by solving Maxwell's equations is a central task in many geophysical applications. In most cases, geophysical targets of interest exhibit complex topography and bathymetry as well as layers and faults. Capturing these effects accurately in numerical simulations is challenging. Standard approaches require a very fine discretization that can result in an impracticably large linear system to be solved. A remedy is to use locally refined and adaptive meshes, however, the potential coarsening is limited in the presence of highly heterogeneous and anisotropic conductivities.
In this paper we discuss the application of Multiscale Finite Volume (MSFV) methods to Maxwell's equations in frequency domain. Given a partition of the fine mesh into a coarse mesh the idea is to obtain coarse-to-fine interpolation by solving local versions of Maxwell's equations on each coarsened grid cell. By construction, the interpolation accounts for fine scale conductivity changes, yields a natural homogenization, and reduces the fine mesh problem dramatically in size. We show that using MSFV methods we can simulate electromagnetic fields with reasonable accuracy in a fraction of the time as compared to state-of-the-art solvers for the fine mesh problem, especially when considering parallel platforms.

INTRODUCTION

Simulating electromagnetic fields by solving Maxwell's equations is a central task in many geophysical applications; see, for example (Ward and Hohmann (1988); Weaver (1994); Newman and Alumbaugh (1995); Hyman and Shashkov (1999a); Monk (2003); Weiss and Newman (2003); Hu et al. (2006)).

Numerical and computational challenges in solving Maxwell's equations often arise from the complexity of geological targets and the experimental setup. In most geophysical applications the region of interest exhibits significant topography and bathymetry as well as layers and faults. Additional complications may arise from large or infinite domains, large scale conductors positioned far away from the region of interest, and varying skin depth effects of electromagnetic fields in media of heterogeneous and anisotropic conductivity.

Capturing the above effects in numerical simulations with a sufficient level of detail generally requires a discretization of Maxwell's equations on a relatively fine mesh and results in an impractically large linear systems to be solved. Especially at low frequencies and domains with highly heterogeneous conductivities these systems are known to be highly ill-conditioned, limiting the performance of iterative solvers; see experiments of Hu et al. (2006).

Recently, adaptive and locally refined meshes have been used successfully to model even complex geometries and accommodate a wide range of experimental setups with relatively few degrees of freedom; see, for instance, (Lipnikov et al. (2004); Schwarzbach (2009); Key and Ovall (2011); Horesh and Haber (2011)). These method have been shown to achieve reasonable accuracy at drastically reduced computational costs. A difficulty in standard mesh refinement techniques is, however, to accurately model heterogeneous conductivities without resorting to locally fine discretizations.

In this work we present a Multiscale Finite Volume (MSFV) method for solving Maxwell's equations at low frequencies. Our goal is to derive a method that yields accurate solutions using very coarse meshes even when the conductivity is highly heterogeneous. To our best knowledge, MSFV methods have not been applied to Maxwell's equations. However, for other problems such as flow in porous media, they have been shown to lead to substantial savings over other finite volume or element methods; see, for instance, (Hou and Wu (1997); Jenny et al. (2003); Hajibeygi et al. (2008, 2009)).

The starting point for our MSFV method is a partition of the fine mesh into coarse grid cells. The centerpiece of the MSFV method is an operator-induced interpolation from the coarse mesh to the fine mesh. To this end, we solve Maxwell's equations *locally* on each coarse mesh cell with specialized boundary conditions and/or source terms. By this construction the interpolation captures fine scale changes of conductivity and yields a natural homogenization of the fine mesh problem. The reduced problem can typically be solved easily and finally applying the interpolation yields an approximate fine mesh solution.

This paper is organized as follows: In the next section we review mimetic finite volume methods for Maxwell's equations in frequency domain and describe the proposed MSFV method. Thereafter, we illustrate the potential of our method on an airborne electromagnetic experiment. Multiscale methods are particularly appealing for this class of problems, since although the solution is required only in a small region around the measurement location it can be influenced considerably by far away topographical features. We show that a locally coarsened MSFV method gives reasonable results at dramatically reduced costs as compared to a state-of-the-art iterative method applied to the fine mesh problem.

MULTISCALE FINITE VOLUME METHOD

Maxwell's equations in frequency domain

We consider the solution of Maxwell's equations in frequency

DOI http://dx.doi.org/10.1190/segam2014-1654.1

domain

$$\begin{aligned} \vec{\nabla} \times \vec{E} + \imath\omega\vec{B} &= 0, \\ \vec{\nabla} \times \mu^{-1}\vec{B} - \sigma\vec{E} &= \vec{s} \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, μ the magnetic permeability, σ, the conductivity, $\vec{s}$ the forcing term, ω is the frequency, and $\Omega \subset \mathbb{R}^3$ is a domain. We are particularly interested in the case that the conductivity, σ, is anisotropic and highly heterogeneous.

For simplicity we restrict the following discussion to the so-called *natural* boundary conditions

$$\vec{\nabla} \times \vec{E} \times \vec{n} = 0 \quad \text{or} \quad \vec{n} \times \vec{E} = 0 \quad \text{on } \partial\Omega. \tag{2}$$

For different boundary conditions corrections are to be made as described in detail by Jin (1993) and Hyman and Shashkov (1999a).

To prepare the description of the mimetic finite volume discretization in the next section, we write (1) in weak form

$$\begin{aligned} (\vec{F}, \vec{\nabla} \times \vec{E}) + \imath\omega(\vec{F}, \vec{B}) &= 0, \\ (\mu^{-1}\vec{B}, \vec{\nabla} \times \vec{W}) - (\sigma\vec{E}, \vec{W}) &= (\vec{s}, \vec{W}). \end{aligned} \tag{3}$$

Here, $\vec{F}$ and $\vec{W}$ are test functions that are in the same function space as $\vec{B}$ and $\vec{E}$, respectively, and $(\cdot,\cdot)$ denotes the $\mathscr{L}_2$ inner product, for example,

$$(\vec{F}, \vec{B}) = \int_\Omega \vec{F}^\top \cdot \vec{B}\, d\vec{x}.$$

Note that we integrated Ampère's law by parts and eliminated the boundary term using the fact that $\vec{W}$ satisfies the same boundary conditions imposed on $\vec{E}$ in (2).

Mimetic Finite Volume Method

We now briefly review the Mimetic Finite Volume (MFV) discretization of (3) as initially proposed by Hyman and Shashkov (1998, 1999a,b). The method is *mimetic* in the sense that solutions to the discretized problem mimic important properties of continuous solutions, for instance, discrete divergence-free properties and conservation of electromagnetic energy are guaranteed even for non-orthogonal meshes and heterogeneous conductivities as shown by Hyman and Shashkov (1999a).

For simplicity we assume that the domain, Ω, is rectangular and is discretized by using an orthogonal tensor mesh, $\mathscr{S}$. As suggested by Hyman and Shashkov (1999a), we represent the electric field $\vec{E} = [\vec{E}_x, \vec{E}_y, \vec{E}_z]$ by the grid functions $\mathbf{e} = [\mathbf{e}_x^\top, \mathbf{e}_y^\top, \mathbf{e}_z^\top]^\top$ which are the orthogonal projections of $\vec{E}$ onto the directions of the edges of the mesh. Similarly we represent the magnetic field $\vec{B} = [\vec{B}_x, \vec{B}_y, \vec{B}_z]$ by the grid functions $\mathbf{b} = [\mathbf{b}_x^\top, \mathbf{b}_y^\top, \mathbf{b}_z^\top]^\top$ which are the orthogonal projection of $\vec{B}$ onto the face normals. We like to emphasize, however, that the following discretization easily extends to unstructured and non-uniform meshes; see (Hyman and Shashkov (1999a)).

To obtain an approximation to the **curl** of the electric field on surfaces of our computational mesh we use Stokes' Theorem and approximate the line integrals using a midpoint quadrature rule on each edge. This yields

$$\vec{\nabla} \times \vec{E} \approx \mathsf{CURL}\ \mathbf{e} = \mathbf{S}^{-1}\ \mathbf{C}\ \mathbf{L}\ \mathbf{e},$$

where $\mathbf{S}$ is a diagonal matrix that contains the area of each face in our mesh, $\mathbf{L}$ is a diagonal matrix that contains the length of each edge in our mesh, and $\mathbf{C}$ is a matrix that contains the values 0 and ± 1 that encodes the mesh connectivity.

Our discretization is completed by discretizing the inner products

$$(\sigma\vec{E}, \vec{W}) \quad \text{and} \quad (\vec{F}, \mu^{-1}\vec{B})$$

of edge-staggered and face-staggered grid functions, respectively. Note that in both cases different components of the vector fields are discretized at different locations and thus cannot be added directly. Here, we use next neighbor interpolations from edges or faces to the eight nodes of each voxel, respectively. Subsequently, we use a trapezoidal rule to approximate the inner products. This yields symmetric mass matrices $\mathbf{M}_e(\sigma)$ and $\mathbf{M}_e(\mu^{-1})$ such that

$$(\sigma\vec{E}, \vec{W}) \approx \mathbf{e}^\top \mathbf{M}_e(\sigma)\mathbf{w} \quad \text{and} \quad (\vec{F}, \mu^{-1}\vec{B}) \approx \mathbf{f}^\top \mathbf{M}_f(\mu^{-1})\mathbf{b}.$$

In summary, we obtain the following discrete analog to Maxwell's equations

$$\begin{aligned} \mathsf{CURL}\ \mathbf{e} + \imath\omega\mathbf{b} &= 0, \\ \mathsf{CURL}^\top \mathbf{M}_f(\mu^{-1})\ \mathbf{b} - \mathbf{M}_e(\sigma)\ \mathbf{e} &= \mathbf{s}, \end{aligned} \tag{4}$$

which can be solved for $\mathbf{e}$ and $\mathbf{b}$. It is common to eliminate $\mathbf{b}$ and obtain a system for $\mathbf{e}$ alone

$$\mathbf{A}(\sigma)\mathbf{e} = \left(\mathsf{CURL}^\top \mathbf{M}_f(\mu^{-1})\mathsf{CURL} - \imath\omega\mathbf{M}_e(\sigma)\right)\mathbf{e} = -\imath\omega\mathbf{s}$$

Depending on the size of the mesh, this sparse linear system can be solved using a direct or iterative method to obtain a discrete solution $\mathbf{e}$.

Multiscale Method

We now present the main idea behind the multiscale method and refer to Jenny et al. (2003) and Efendiev and Hou (2009) for more details. Our adaption of multiscale methods to Maxwell's equations follows Efendiev and Hou (2009), however, changes arise from the edge staggered discretization.

We consider a fine 3D tensor mesh, $\mathscr{S}_h$ with n cells, and a partition into a coarse mesh $\mathscr{S}_H$ with N cells, where $N \ll n$.

The centerpiece of our method is an interpolation operator $\mathbf{E}$ from the edges of $\mathscr{S}_H$ to the edges of $\mathscr{S}_h$. Using $\mathbf{E}$, the fine mesh problem

$$\mathbf{A}_h(\sigma)\ \mathbf{e}_h = -\imath\omega\ \mathbf{s} = \mathbf{q}$$

is projected to the coarse mesh where it can be solved easily yielding an approximation

$$\mathbf{e}_{\text{MS}} = \mathbf{E}\ (\mathbf{E}^\top \mathbf{A}_h(\sigma)\mathbf{E})^{-1}\mathbf{E}^\top \mathbf{q}.$$

The quality of this approximation depends on how well the fine scale solution, $\mathbf{e}_h$, and the right hand side, $\mathbf{q}$, can be described in the low-dimensional subspace spanned by the columns of $\mathbf{E}$. The computational savings depends on the ratio of the number of edges in the coarse mesh versus the fine mesh.

For illustration, we first introduce a simple example for the interpolation. Let us consider an arbitrary coarse mesh cell Ω_H^j, for $j = 1, \ldots, N$ and its associated i-th edge. We may define

DOI http://dx.doi.org/10.1190/segam2014-1654.1

a local interpolation using a basis function $\phi_i^j : \Omega_H^j \to \mathbb{R}^3$ that is oriented and constant along the direction of the edge and decays linearly in the other two directions. In our application this straightforward and inexpensive approach is natural for $\omega \to 0$ and homogeneous coefficients and thus commonly used in the absence of any other criteria. However, large errors may result if the conductivity, σ, or the right hand side, $\mathbf{q}$, vary considerably within the coarse mesh cell.

Multiscale methods aim to better capture fine scale information by constructing an *operator-induced* interpolation. Considering as above the i-th edge of the j-th coarse mesh cell, an interpolation of the electric field $\vec{E}_i^j$ is obtained as the solution of

$$\begin{aligned} \vec{\nabla} \times \vec{E}_i^j + \imath\omega \vec{B}_i^j &= 0, \\ \vec{\nabla} \times \mu^{-1} \vec{B}_i^j - \sigma \vec{E}_i^j &= 0 \qquad \text{in } \Omega_H^j, \\ \vec{n} \times \vec{E}_i^j &= \vec{n} \times \phi_i^j \quad \text{on } \partial\Omega_H^j. \end{aligned} \tag{5}$$

Note that the operator-induced interpolation $\vec{E}_i^j$ and the basis function ϕ_i^j in general only agree on the boundary. Note also that while ϕ_i^j interpolates each component by itself, in $\vec{E}_i^j$ there is *leakage of components*, that is, generally all three components of $\vec{E}_i^j$ will be non-zero. The interpolation operator $\mathbf{E}$ is obtained by repeating this process for all edges and all coarse mesh cells.

We have found in our numerical experiments that the interpolation obtained from (5) does not always give sufficient accuracy when using a very coarse mesh. As also suggested in Efendiev and Hou (2009) a remedy is to augment the interpolation matrix, $\mathbf{E}$, by a few basis functions that depend on the source, $\vec{s}$. Without loss of generality, we assume that the restriction of $\vec{s}$ to Ω_H^j, denoted by $\vec{s}^j$, does not vanish and consider

$$\begin{aligned} \vec{\nabla} \times \vec{E}_s^j + \imath\omega \vec{B}_s^j &= 0, \\ \vec{\nabla} \times \mu^{-1} \vec{B}_s^j - \sigma \vec{E}_s^j &= \vec{s}^j \quad \text{in } \Omega_H^j, \\ \vec{n} \times \vec{E}_s^j &= 0 \quad \text{on } \partial\Omega_H^j. \end{aligned} \tag{6}$$

This approach is particularly attractive when the source, $\vec{s}$, is highly localized and affects only a small number of coarse mesh cells and thus only few degrees of freedom are added. This assumption is valid in geophysical applications in which $\vec{s}$ is a combination of delta functions, for instance, a loop source.

For each cell in the coarse mesh, $\mathscr{S}_H$, problems (5) and (6) are discretized using the respective part of the fine mesh, $\mathscr{S}_h$, by applying the mimetic finite volume method outlined above. If the number of fine mesh cells contained in a coarse mesh cell is relatively small, for instance, smaller than some number $m \ll n$, the local problems can be solved quickly using a direct method in $\mathscr{O}(m^3)$ operations. This clearly is more expensive than constructing an operator-independent interpolation. However, it is important to note that this computation is trivially parallel and therefore can be done efficiently given a parallel platform.

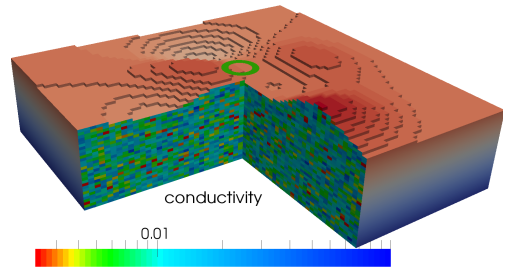

Figure 1: Setup for our numerical experiment. A magnetic loop of size 50 m, represented by a green circle, is placed in the air above media with random anisotropic conductivities, here represented by the $\sigma_{1,1}$ component.

AIRBORNE ELECTROMAGNETIC EXPERIMENT

We demonstrate the effectiveness of the proposed Multiscale Finite Volume method for Maxwell's equations based on a test problem mimicking a common geophysical survey where a magnetic loop of size 50 m is placed above a non-homogeneous medium. Our setting is similar to the other airborne electromagnetic experiments such as, for instance, the one considered by Viezzoli et al. (2009). As common in this setting, the magnetic loop is used both as a transmitter and a receiver.

We assume that the conductivity is made from large topographical features that range from -300 m to 300 m above the surface and that the magnetic loop is placed near the topographical features. We use an anisotropic conductivity tensor whose components inside the earth are assumed to be distributed $\log_{10}$ normally with mean of -2 and variance of $1/4$. The conductivity inside the earth ranges from roughly 10^{-2} S/m to approximately 10^{-4} S/m. A sketch of our experiment is visualized in Figure 1. We test the effectiveness of the scheme for three frequencies $\omega = 10, 100, 1000$ Hz. Note that for 1000 Hz the skin depth is roughly 175 m which implies that we need 50 m cell size to capture the decaying nature of the electric fields.

The fine mesh consists of 72^3 cells and is centered around the source. The region around the source is discretized using a regular mesh with 60^3 cells of width 50 m. To reduce the impact of boundary conditions, we pad the mesh by adding 6 cells in each direction whose edge width is increased by a factor of 1.3 towards the boundary. Discretizing Maxwell's equations as outlined above, yields a fine mesh problem with roughly 1.1 million degrees of freedom.

Since there is no analytic solution to this problem, we compute a reference solution using a state-of-the-art iterative solver. To be precisely, we use the biconjugate gradient stabilized method (bicgstab) and the preconditioner suggested by Haber and Heldmann (2007) on a standard desktop computer using MATLAB. This yields reference solution with relative residuals of $4.5 \cdot 10^{-9}$, $4.9 \cdot 10^{-10}$, and $4.2 \cdot 10^{-10}$ for the respective frequen-

DOI http://dx.doi.org/10.1190/segam2014-1654.1

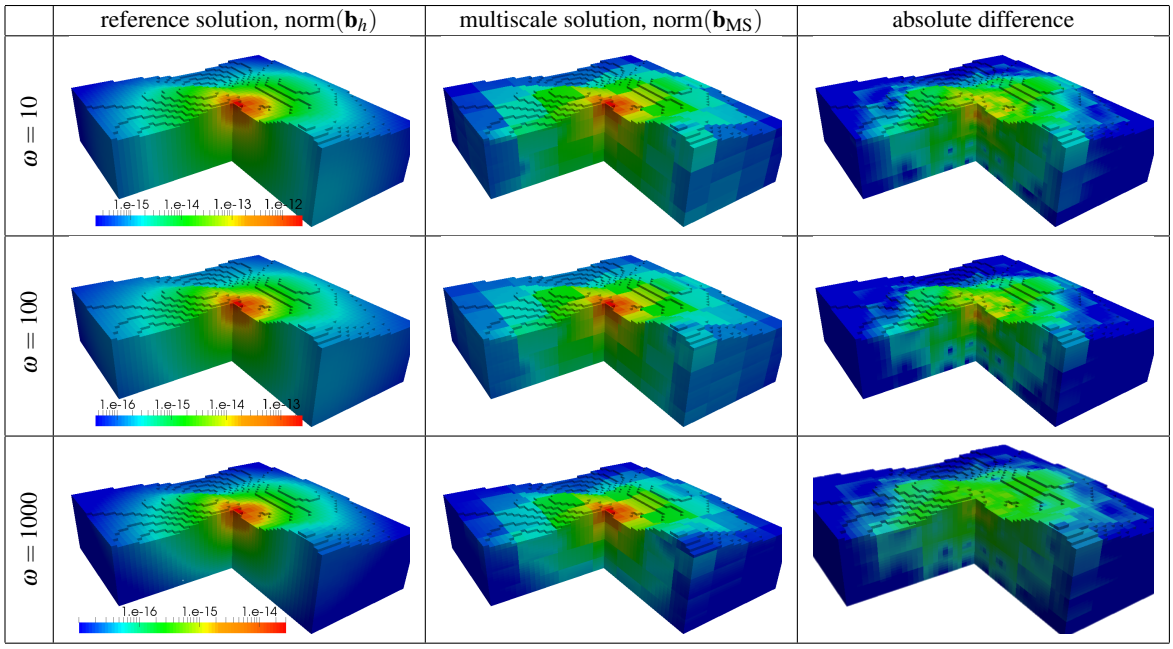

Figure 2: Numerical solutions for the airborne electromagnetic experiment at frequencies $\omega = 10, 100, 1000$ (row-wise). The first column visualizes the magnetic field strength computed by solving the problem on the fine mesh, a tensor mesh with 72^3 cells, around the location of the loop source. The middle column shows the magnetic field obtained by proposed multiscale finite volume method and the right column the absolute differences. All fields are shown on a logarithmic scale and to enable comparison the same colour axis is chosen in each row.

cies. The absolute values of the magnetic fields are visualized around the source location in the left column of Figure 2.

The coarse base mesh consists of 9^3 cells with 8^3 fine mesh cells each. For a completely regular partition, the loop source would be located on the faces of the coarse mesh and large errors have to be expected. To reduce inaccuracies arising from the interaction of the source with the coarse mesh faces, we do not coarsen cells that are connected to the source. Following the multiscale strategy outlined above, we reduce the size of the fine mesh problem to only 5,164 degrees of freedom. The absolute value of the magnetic fields computed using the multiscale method are visualized in the middle column of Figure 2.

We now compare the results obtained using the presented multiscale method to results obtained using the fine mesh problem. The relative errors between the computed magnetic fields are around 4.1% for all frequencies. The real parts of simulated measurements inside the loop deviate by around 0.2% while the variation in the imaginary part is less than 0.8% for all frequencies. However, even in the current implementation using MATLAB and without fully exploiting the parallel nature of the algorithm the multiscale method requires around 150 seconds to build the interpolation and less than a second to solve the reduced problems.

CONCLUSIONS

In this work we presented a multiscale finite volume method for solving Maxwell's equations with heterogeneous conductivities at low frequencies.

The centerpiece of the method is generating an operator-induced interpolation from edges of the coarse mesh to edges of the fine mesh. To this end, we solve Maxwell's equations *locally* on every coarse cell mesh yielding a large number of small and easy-to-solve subproblems. While this process is more expensive than using an operator-independent interpolation, for instance, using piecewise linear and constant basis functions, the multiscale interpolation can capture fine scale heterogeneity of conductivity within a coarse mesh cell more accurately and thus potentially allows for larger coarsening in many applications. Furthermore the interpolation can be constructed independently for all coarse mesh cells and is thus a good candidate for parallel implementation.

We demonstrated the potential of the MSFV method using an airborne electromagnetic experiment. For this setup the method yields comparable results at heavily reduced costs compared to the fine mesh problem. The results are remarkable given the large ratio between skin depth and size of the coarse mesh cells and the presence of large conductivity variations.

Our results are in line with those reported in other areas of application of multiscale methods. However, we emphasize the MSFV method is more appealing for Maxwell's equations than, for example, for Poisson' equation. While the use of MSFV methods for Poisson's equation is mainly dealing with heterogeneity, in our context they are also promising for homogeneous media when the skin depth requires a fine discretization.

DOI http://dx.doi.org/10.1190/segam2014-1654.1

http://dx.doi.org/10.1190/segam2014-1654.1

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